Applied Mathematics and Artificial Intelligence: Second-round sample tasks for the Open Doors Postgraduate Track

You will be asked to complete 30 tasks, including:

- 19 entry-level tasks, each correct answer worth 1-3 points;
- 8 intermediate-level tasks, each correctly answered task worth 3-7 points;
- 3 advanced tasks (constructed response), each correctly completed task valued at 5–15 points.

Evaluation criteria and standard answers are provided for the advanced tasks requiring constructed responses.

Mathematics

Task 1
Entry-level (2 points)

Find the distance between two straight lines l_1 and l_2.

$$l_1: \begin{cases} x = t \\ y = t \\ z = t \end{cases} \quad t \in \mathbb{R} \qquad\qquad l_2: \begin{cases} x = 1 \\ y = 2 \\ z = t \end{cases} \quad t \in \mathbb{R}$$

a) $\frac{1}{\sqrt{2}}$
b) $\frac{1}{\sqrt{3}}$
c) 0
d) $\sqrt{2}$

Answer: a.

Task 2
Entry-level (1 points)

Consider $U \subset \mathbb{R}^4$: $U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\rangle$. Find $\dim U$.

a) 1
b) 2
c) 3
d) 4

Answer: c.

Task 3
Intermediate-level (5 points)

Reduce the matrix of the linear transformation $\begin{pmatrix} 3 & 1 & 3 & 4 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & -3 & -1 \end{pmatrix}$ to the Jordan normal form. In the answer, write down the Jordan normal form of the matrix.

Answer: $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Task 4
Entry-level (3 points)

Evaluate the limit: $\lim_{n\to\infty} \ln\left(\sqrt{9n^2+18n}-3n\right)$.

a) ln3
b) 0
c) ln6
d) there is no limit

Answer: a.

Task 5
Entry-level (3 points)

Evaluate the integral: $\int_{-\pi}^{\pi}(x^6+x)\cdot\sin x\,d\,x$.

a) 0
b) π
c) 2π
d) −2π

Answer: c.

Task 6
Advanced (8 points)

In the linear space of functions continuous on the interval [-π;π], the scalar product of the elements $(f,g)=\int_{-\pi}^{\pi}f(x)g(x)dx$ and the distance between the elements $\rho(f,g)=\sqrt{(f-g,f-g)}$ are given. Find the distance from the function $f(x) = x$ to the subspace L, where L is the linear span of the functions $f_0(x) = 1$, $f_1(x) = \sin x$, $f_2(x) = \cos x$.

Solution:
Let us denote the space of functions continuous on the interval $[-\pi; \pi]$ by E. We expand E into the direct sum of L and $L^\perp$: $E = L \oplus L^\perp$. Then $x = y + z$, where $y \in L$, and $z \in L^\perp$. By definition, $\rho(x, L) = \min_{y \in L} \rho(x, y)$. It is known that $\min_{y \in L} \rho(x, y) = \rho(z, 0)$. Let us find z. We expand y in the basis of L: $y = a_0 \cdot 1 + a_1 \cdot \sin x + a_2 \cdot \cos x$. We have $x = a_0 \cdot 1 + a_1 \cdot \sin x + a_2 \cdot \cos x + z$. Multiplying this equality scalarly by 1, $\sin x$, $\cos x$, we get $x = 2 \cdot \sin x + z$. Therefore, $z = x - 2 \cdot \sin x$, and $\rho(z, 0) = \sqrt{\int_{-\pi}^{\pi} (x - 2\sin x)^2 dx}$. We calculate

$$\int_{-\pi}^{\pi} (x - 2\sin x)^2 dx = \int_{-\pi}^{\pi} (x^2 - 4x\sin x + 4\sin^2 x) dx = 2\pi \left(\frac{\pi^2}{3} - 2\right).$$

As a result, we get $\rho(x, L) = \sqrt{2\pi\left(\frac{\pi^2}{3} - 2\right)}$.

Answer: $\rho(x, L) = \sqrt{2\pi\left(\frac{\pi^2}{3} - 2\right)}$.

Assessment criteria:
The answer was given without justification - 0 points.
There are correct ideas regarding the solution to the problem - 2 points.
The orthogonal projection of the function $f(x) = x$ onto L is found - 4 points.
The problem of calculating the desired distance is reduced to calculating the orthogonal component - 6 points.
The problem is completely solved - 8 points.

Applied mathematics

Task 7
Entry-level (3 points)

Find the greatest common divisor of $x^3 + x^2 - 2$ and $x^3 + 2x^2 + 2x$.

a) $x^2 + 2x + 2$
b) x
c) $x - 1$
d) 1

Answer: a.

Task 8
Intermediate-level (5 points)

Find the number of integers x, $x \in [0; 100]$, for which $x^9 + 1$ is divisible by 15.

Answer: 6.

Task 9
Entry-level (3 points)

The functions 2, $x + 2$, $x^2 - 2$ are solutions of the equation $y'' + a(x)y' + b(x)y = c(x)$. Find $a(1)$.

a) -0.8
b) -0.4
c) -1
d) 0

Answer: b

Task 10
Advanced (8 points)

The phase trajectory of the system $\begin{cases} \frac{dx}{dt} = 2x - 5y, \\ \frac{dy}{dt} = 8x - 2y, \end{cases}$ passes through the point (–1;2). Find the maximum distance from the points of this phase trajectory to the point (0;0).

Solution:

Let us write down the matrix of the system $\begin{pmatrix} 2 & -5 \\ 8 & -2 \end{pmatrix}$ and find its eigenvalues: $\lambda_1 = -6i$, $\lambda_2 = 6i$. Therefore, the point (0;0) is the centre, and the phase trajectories are ellipses with the centre at the point (0;0). These trajectories are solutions of the equation $\frac{dy}{dx} = \frac{8x-2y}{2x-5y}$. Let us write this equation in the symmetric form: $(8x - 2y)dx + (5y - 2x)dy = 0$. Rewrite it as follows: $d\left(4x^2 - 2xy + \frac{5}{2}y^2\right) = 0$. Its general solution is $8x^2 - 4xy + 5y^2 = C$. Let's substitute the point (–1;2) into this equation and find C: $8 + 4 + 20 = C$. As a result, we obtain the phase trajectory equation: $8x^2 - 4xy + 5y^2 = 36$. This is the ellipse. We need to find the canonical form of this ellipse. The eigenvalues of a matrix $\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$ of quadratic form $8x^2 - 4xy + 5y^2$ are equal to 4 and 9. The canonical equation of the ellipse is $4\tilde{x}^2 + 9\tilde{y}^2 = 36$ or $\frac{\tilde{x}^2}{9} + \frac{\tilde{y}^2}{4} = 1$. The semi-axes of the ellipse are equal to 3 and 2. Therefore, the maximum distance from the points of the ellipse to the point (0;0) is equal to 3.

Answer: 3.

Assessment criteria:

The answer was given without justification - 0 points.
A family of phase trajectories of the system is obtained - 2 points.
The phase trajectory passing through the given point is obtained - 4 points.
The canonical equation of the ellipse is obtained - 6 points.
The problem is completely solved - 8 points.

Mathematical Physics

Task 11
Entry-level (1 point)

Which of the functions is harmonic in the domain $x^2 + y^2 \leq 4$?

a) $x^2 - y^2$
b) $x^3 + 3y$
c) $x^2 + y^2$
d) $1/\sqrt{x^2 + y^2}$

Answer: a.

Task 12
Entry-level (2 points)

Determine the type of second-order partial differential equation with respect to a function of two variables $u(x,\ y)$: $u_{xx} + 2 \cdot u_{xy} - 9 \cdot u_{yy} = 0$.

a) hyperbolic
b) parabolic
c) elliptical

Answer: a.

Task 13
Intermediate-level (4 points)

Find the eigenvalues of the Sturm-Liouville problem on the interval $\left(\frac{31}{5}; \frac{25}{2}\right)$:

$$\begin{cases} Y''(x) + \lambda \cdot Y(x) = 0, \\ Y(0) = Y(\pi) = 0. \end{cases}$$

Answer: 9.

Task 14
Intermediate-level (5 points)

Find a solution to the Cauchy problem $\begin{cases} u_{xx} - u_{tt} = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = 0, u_t(x,0) = \sin x. \end{cases}$

Answer: $u(x,t) = \sin x \cdot \sin t$.

Statistics & probability

Task 15
Entry-level (2 points)

A point was randomly placed on the segment from 2 to 5. What is the probability that this point fell in the interval between 3 and 4?

a) $\frac{1}{3}$
b) $\frac{3}{10}$
c) $\frac{1}{2}$
d) $\frac{1}{5}$

Answer: a.

Task 16
Entry-level (3 points)

The random variable has a normal distribution. Distribution density is

$$p(x) = \frac{1}{\sqrt{6\pi}} e^{-\frac{(t-6)^2}{9}}.$$

Find the mathematical expectation of the random variable.

a) 6
b) 0
c) 3
d) 9

Answer: a.

Task 17
Intermediate-level (4 points)

The joint density of the distribution of random variables ξ and η is given:

$$p_{\xi\eta}(x,y) = \begin{cases} 1/4\pi, & x^2 + y^2 \le 4, \\ 0, & \text{in other cases.} \end{cases}$$

Find the correlation coefficient $r(\xi,\eta)$. Are the random variables dependent?

Answer: $r(\xi,\eta) = 0$. The random variables are dependent.

Task 18
Advanced (6 points)

From a large number of prototypes, to continue the experiment, a laboratory technician must select two with a certain property. The probability of a sample having this property is $p = 0{,}2$. How many samples should the laboratory technician check on average to select the required ones?

Solution:
Let ξ be the number of tested prototypes. Then $P\{\xi = k\} = C_{k-1}^{1} p^2 (1-p)^{k-2}$.

$$M\xi = \sum_{k=2}^{\infty} k \cdot P\{\xi = k\} = \sum_{k=2}^{\infty} k(k-1)\, p^2 (1-p)^{k-2} = p^2 \sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2}$$

Note that

$$\sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{d^2}{dx^2}\sum_{k=2}^{\infty} x^k = \frac{d^2}{dx^2}\left(\frac{x^2}{1-x}\right) = \frac{2}{(1-x)^3}$$

Then the average value of the tested samples is

$$M\xi = p^2 \frac{2}{\left(1-(1-p)\right)^3} = \frac{2}{p} = 10$$

Answer: $M\xi = 10$.

Assessment criteria:
The answer was given without justification - 0 points.
You have the right ideas about how to solve the problem - 2 points.
The distribution law of the random variable was correctly written out - 4 points.
A formula for calculating the expected value of the random variable has been correctly written - 6 points.
The problem is completely solved -8 points.

Computer science, artificial intelligence

Task 19
Entry-level (2 points)

Let (0, 1), (1, 1), (2, 5) be a training sample, each element is (x, y), where x is the input variable, y is the target output. Find the coefficient of determination of the simplest linear regression model y = ax + b, trained on this sample using the ordinary least squares method.

a) 0.75

b) 0.866
c) 0.6667
d) 0.725

Answer: a.

Task 20
Entry-level (2 points)

On some datasets, the classifier predicted the following class labels: 1, 2, 1, 3, 3, 2, 2, 3. The corresponding target labels are 1, 3, 3, 3, 1, 2, 2, 1. Find the macro-averaged F1-score of the classifier on this dataset.

a) 0.5111
b) 0.5556
c) 0.5
d) 0.475

Answer: a.

Task 21
Entry-level (3 points)

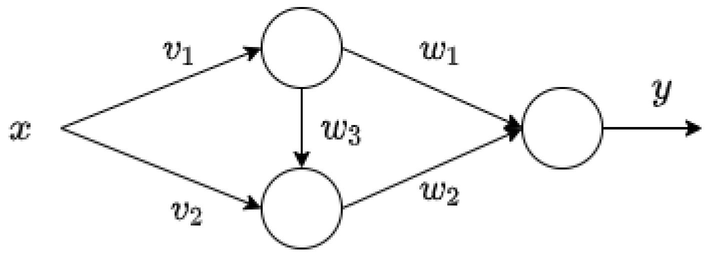
Three trained classifiers assign the input example to the correct class with probabilities 0.8, 0.9 and 0.75, respectively. Assuming that the errors of the classifiers are independent, determine the probability of correct classification by an ensemble of these classifiers with a simple majority voting.

a) 0.915
b) 0.8167
c) 0.85
d) 0.9

Answer: a.

Task 22
Intermediate-level (5 points)

A neural network shown in the figure below is trained by stochastic gradient descent with a learning rate $\alpha = 0.1$.



The output neuron's transfer function is linear, and the other neurons have the transfer function defined as:

$$f(h) = \begin{cases} h, & h \geq 0, \\ 0, & h < 0. \end{cases}$$

At the current training step the synaptic coefficients (weights) in the network have taken the values $v_1 = v_2 = w_1 = w_2 = 1$, $w_3 = -2$, and the biases of all neurons are zero. The network receives a training example $x = 2$ with the target value $\sigma = 3$. Perform a network's training step with a quadratic loss function $L = \frac{1}{2}(\sigma - y)^2$ and calculate the synaptic coefficient v_1 after its adjustment.

Answer: 1.2.

Computer science, cybernetics

Task 23
Entry-level (1 point)

The slope of the high-frequency part of the asymptotic logarithmic amplitude-frequency response of the aperiodic link is equal to

a) –20 dB/dec.
b) 0 dB/dec.
c) 20 dB/dec.
d) 40 dB/dec.
e) –40 dB/dec.

Answer: a.

Task 24
Entry-level (2 points)

The transfer function of the system is $W(s) = \frac{s^2-2s+4}{s^2-25}$. Which typical dynamic link does it contain?

a) unstable aperiodic
b) first-order differentiator

c) second-order differentiator
d) oscillatory
e) integrating

Answer: a.

Task 25
Entry-level (3 points)

Determine the duration of the transition process in a nonlinear system from point A to point D, if the corresponding section of the phase trajectory on the phase plane xoy (x is the output coordinate of the system, y is the rate of its change) consists of straight segments connecting points A(–4;3), B(2;3), C(2;1) and D(6;1).

a) 6
b) 2
c) 4
d) 8

Answer: a.

Task 26
Intermediate-level (4 points)

At what values of parameter "a" a closed-loop system (single and negative feedback) is located on the boundary of its stability, if the differential equation of this system in an open-loop state has the following form (here u(t) and y(t) are the input and output of the open-loop system, respectively):

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} - ay(t) = \frac{d^2u(t)}{dt^2} - a\frac{du(t)}{dt} + 5u(t).$$

Answer: 4.

Computer science, information systems

Task 27
Entry-level (1 point)

What is the main structural part of the relational model?

a) attitude
b) attribute
c) tuple

Answer: a.

Task 28
Entry-level (1 point)

According to the CAP theorem, how many properties can be satisfied in a distributed system?

a) 1
b) 2
c) 3
d) 4

Answer: b.

Task 29
Entry-level (2 points)

What are ETL processes used for?

a) to ensure quality, create the necessary structure and maintain the semantic characteristics of the data
b) to create relational tables
c) to carry out procedures for searching, filtering and sorting data in the warehouse after receiving an SQL query

Answer: a.

Task 30
Intermediate-level (4 points)

There are two relations given - A and B. Relation A has the form {Student ID, Student Last Name, Group, Grade}, and relation B has the form {Student ID, Student Last Name, Student Name, Subject, Grade}. How many attributes will there be in relation C = A JOIN B?

Answer: 6.