Applied Mathematics and Artificial Intelligence: Second-round Sample Tasks for the Open Doors Master's and Doctoral Track

This sample test comprises 30 tasks, including: 19 entry-level tasks with a single correct answer, each correct answer assigned 1 to 3 points; 8 intermediate-level tasks with multiple correct answers, each correct answer is assigned 3 to 7 points; 3 advanced-level tasks requiring a detailed answer, each answer is assigned 5 to 15 points, depending on its correctness and completeness.
For advanced-level tasks requiring a detailed answer, assessment criteria and a standard answer are provided.

Field of Science 1. Mathematics

Task 1
Entry level (2 points)

Find the distance between two straight lines l_1 and l_2.

$$l_1: \begin{cases} x = t \\ y = t \\ z = t \end{cases} \quad t \in R \qquad\qquad l_2: \begin{cases} x = 1 \\ y = 2 \\ z = t \end{cases} \quad t \in R$$

a) $\frac{1}{\sqrt{2}}$
b) $\frac{1}{\sqrt{3}}$
c) 0
d) $\sqrt{2}$

Answer: a.

Task 2
Entry level (1 points)

Consider $U \subset R^4$: $U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\rangle$. Find $\dim U$.

a) 1
b) 2
c) 3
d) 4

Answer: c.

Task 3
Intermediate level (5 points)

Reduce the matrix of the linear transformation $\begin{pmatrix} 3 & 1 & 3 & 4 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & -3 & -1 \end{pmatrix}$ to its Jordan normal form. In your answer, provide the Jordan normal form of the matrix.

Answer: $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Task 4
Entry level (3 points)

Evaluate the limit: $\lim\limits_{n\to\infty} \ln\left(\sqrt{9n^2 + 18n} - 3n\right)$.

a) ln3
b) 0
c) ln6
d) There is no limit.

Answer: a.

Task 5
Entry level (3 points)

Evaluate the integral: $\int_{-\pi}^{\pi}(x^6 + x) \cdot \sin x dx$.

a) 0
b) π
c) 2π
d) −2π

Answer: c.

Task 6
Advanced level (8 points)

In the linear space of functions continuous on the interval [−π;π], the scalar product of elements $(f, g) = \int_{-\pi}^{\pi} f(x)g(x)dx$ and the distance between elements $\rho(f, g) = \sqrt{(f - g, f - g)}$ are defined. Find the distance from the function $f(x) = x$ to the subspace L, where L is the linear span of the functions $f_0(x) = 1$, $f_1(x) = \sin x$, $f_2(x) = \cos x$.

Solution:
Let us denote the space of functions continuous on the interval $[-\pi; \pi]$ as E. We decompose E into the direct sum of L and $L^\perp$: $E = L \oplus L^\perp$. Then, $x = y + z$, where $y \in L$, and $z \in L^\perp$. By definition, $\rho(x, L) = \min\limits_{y\in L}\rho(x, y)$. It is known that $\min\limits_{y\in L}\rho(x, y) = \rho(z, 0)$. Let us find z. We expand y in the basis of L: $y = a_0 \cdot 1 + a_1 \cdot \sin x + a_2 \cdot \cos x$. We obtain $x = a_0 \cdot 1 + a_1 \cdot \sin x + a_2 \cdot$

$\cos x + z$. Multiplying this equality scalarly by 1, $\sin x$, $\cos x$, we derive $x = 2 \cdot \sin x + z$. Therefore, $z = x - 2 \cdot \sin x$, and $\rho(z, 0) = \sqrt{\int_{-\pi}^{\pi} (x - 2\sin x)^2 dx}$. Then, we calculate

$$\int_{-\pi}^{\pi} (x - 2\sin x)^2 dx = \int_{-\pi}^{\pi} (x^2 - 4x\sin x + 4\sin^2 x) dx = 2\pi\left(\frac{\pi^2}{3} - 2\right).$$

This yields $\rho(x, L) = \sqrt{2\pi\left(\frac{\pi^2}{3} - 2\right)}$.

Answer: $\rho(x, L) = \sqrt{2\pi\left(\frac{\pi^2}{3} - 2\right)}$.

Assessment criteria

The answer was given without a detailed solution: 0 points.
A viable approach was applied to solving the problem: 2 points.
The orthogonal projection of the function $f(x) = x$ onto L was found: 4 points.
The problem of calculating the desired distance was reduced to finding the orthogonal component: 6 points.
The problem was solved in its entirety: 8 points.

Field of Science 2. Applied Mathematics

Task 7
Entry level (3 points)

Which is the greatest common divisor of $x^3 + x^2 - 2$ and $x^3 + 2x^2 + 2x$.

a) $x^2 + 2x + 2$
b) x
c) $x - 1$
d) 1

Answer: a.

Task 8
Intermediate level (5 points)

Find the number of integers x, $x \in [0; 100]$, for which $x^9 + 1$ is divisible by 15.

Answer: 6.

Task 9
Entry level (3 points)

The functions 2, $x + 2$, $x^2 - 2$ are solutions to the equation $y'' + a(x)y' + b(x)y = c(x)$. Find $a(1)$.

a) -0.8
b) -0.4
c) -1
d) 0

Answer: b

Task 10
Advanced level (8 points)

The phase trajectory of the system $\begin{cases} \frac{dx}{dt} \quad 2x - 5y, \\ \frac{dy}{dt} \quad 8x - 2y, \end{cases}$ passes through the point (–1;2). Find the maximum distance from the points of this phase trajectory to the point (0;0).

Solution:
Let us write down the matrix of the system $\begin{pmatrix} 2 & -5 \\ 8 & -2 \end{pmatrix}$ and find its eigenvalues: $\lambda_1 = -6i$, $\lambda_2 = 6i$. The point (0;0) is the center, and the phase trajectories are ellipses centered at (0;0). These trajectories are solutions to the equation $\frac{dy}{dx} = \frac{8x-2y}{2x-5y}$. Let us transform this equation to symmetric form: $(8x - 2y)dx + (5y - 2x)dy = 0$. Then, we rewrite it as follows: $d\left(4x^2 - 2xy + \frac{5}{2}y^2\right) = 0$. Its general solution is $8x^2 - 4xy + 5y^2 = C$. Let us now substitute the point (–1;2) into this equation and find C: $8 + 4 + 20 = C$. This yields the phase trajectory equation: $8x^2 - 4xy + 5y^2 = 36$, which describes an ellipse. Now, we need to find the canonical form of this ellipse. The eigenvalues of the matrix $\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$ of quadratic form $8x^2 - 4xy + 5y^2$ are equal to 4 and 9. The canonical equation of the ellipse is $4x^2 + 9y^2 = 36$ or $\frac{x^2}{9} + \frac{y^2}{4} = 1$. The semi-axes of the ellipse are equal to 3 and 2. Therefore, the maximum distance from the points of the ellipse to the point (0;0) is 3.

Answer: 3.

Assessment criteria
The answer was given, but no detailed solution was provided: 0 points.
A family of phase trajectories of the system was obtained: 2 points.
The phase trajectory passing through the given point was obtained: 4 points.
The canonical equation of the ellipse was obtained: 6 points.
The problem was solved in its entirety: 8 points.

Field of Science 3. Mathematical Physics

Task 11
Entry level (1 point)

Which of the functions is harmonic in the domain $x^2 + y^2 \leq 4$?

a) $x^2 - y^2$
b) $x^3 + 3y$
c) $x^2 + y^2$
d) $1/\sqrt{x^2 + y^2}$

Answer: a.

Task 12

Entry level (2 points)

Determine what type of second-order partial differential equation applies to the function of two variables u(x, y): $u_{xx} + 2 \cdot u_{xy} - 9 \cdot u_{yy} = 0$.

a) hyperbolic
b) parabolic
c) elliptical

Answer: a.

Task 13
Intermediate level (4 points)

Find the eigenvalues of the Sturm-Liouville problem on the interval $\left(\frac{31}{5}; \frac{25}{2}\right)$:

$$\begin{cases} Y''(x) + \lambda \cdot Y(x) = 0, \\ Y(0) = Y(\pi) = 0. \end{cases}$$

Answer: 9.

Task 14
Intermediate level (5 points)

Find a solution to the Cauchy problem $\begin{cases} u_{xx} - u_{tt} = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = 0, u_t(x,0) = \sin x. \end{cases}$

Answer: $u(x,t) = \sin x \cdot \sin t$.

Field of Science 4. Statistics and Probability

Task 15
Entry level (2 points)

A point was randomly placed on the segment from 2 to 5. What is the probability that this point lies within the interval from 3 to 4?

a) $\frac{1}{3}$
b) $\frac{3}{10}$
c) $\frac{1}{2}$
d) $\frac{1}{5}$

Answer: a.

Task 16
Entry level (3 points)

Find the mathematical expectation of the random variable, if it follows a normal distribution and the distribution density is

$$p(x) = \frac{1}{\sqrt{6\pi}} e^{\frac{-(t-6)^2}{9}}.$$

a) 6
b) 0
c) 3
d) 9

Answer: a.

Task 17
Intermediate level (4 points)

The joint density of the distribution of random variables ξ and η is given:

$$p_{\xi\eta}(x,y) = \begin{cases} \frac{1}{4\pi}, x^2 + y^2 \leq 4, \\ 0,\wedge \text{ othercases.} \end{cases}$$

Find the correlation coefficient $r(\xi,\eta)$ and determine whether the random variables are dependent.

Answer: $r(\xi,\eta) = 0$. The random variables are dependent.

Task 18
Advanced level (6 points)

From a large batch of prototypes, a laboratory technician must select two that possess a specific property in order to continue the experiment. The probability of a sample having this property is $p = 0{,}2$. On average, how many samples must the laboratory technician check to select the required ones?

Solution:
Let ξ be the number of tested prototypes. Then $P\{\xi = k\} = C_{k-1}^{1} p^2 (1-p)^{k-2}$.

$$M\xi = \sum_{k=2}^{\infty} k \cdot P\{\xi = k\} = \sum_{k=2}^{\infty} k(k-1)\, p^2 (1-p)^{k-2} = p^2 \sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2}$$

Note that

$$\sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{d^2}{dx^2} \sum_{k=2}^{\infty} x^k = \frac{d^2}{dx^2}\left(\frac{x^2}{1-x}\right) = \frac{2}{(1-x)^3}$$

Then the average value of the tested samples is

$$M\xi = p^2 \frac{2}{\left(1-(1-p)\right)^3} = \frac{2}{p} = 10$$

Answer: $M\xi = 10$.

Assessment criteria
The answer was given, but no detailed solution was provided: 0 points.
A viable approach was applied to solving the problem: 2 points.
The distribution of the random variable was correctly specified: 4 points.

A correct formula for calculating the expected value of the random variable was provided: 6 points.
The problem was solved in its entirety: 8 points.

Field of Science 5. Artificial Intelligence

Task 19
Entry level (2 points)

Let (0, 1), (1, 1), and (2, 5) be a training sample, where each element is of the form (x, y), with x as the input variable and y as the target output. Find the coefficient of determination for the simplest linear regression model y = ax + b, trained on this sample using the ordinary least squares method.

a) 0.75
b) 0.866
c) 0.6667
d) 0.725

Answer: a.

Task 20
Entry level (2 points)

The classifier predicted the following class labels on a dataset: 1, 2, 1, 3, 3, 2, 2, 3. The corresponding target labels are: 1, 3, 3, 3, 1, 2, 2, 1. Calculate the macro-averaged F1-score of the classifier on this dataset.

a) 0.5111
b) 0.5556
c) 0.5
d) 0.475

Answer: a.

Task 21
Entry level (3 points)

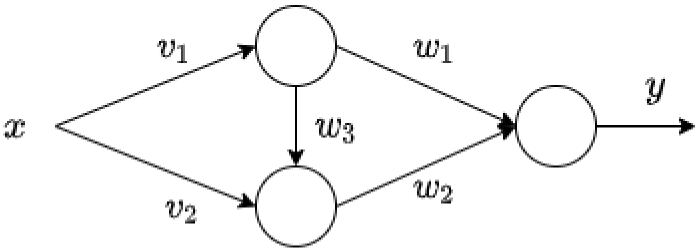
Three trained classifiers assign the input example to the correct class with probabilities of 0.8, 0.9, and 0.75, respectively. Assuming the classifiers' errors are independent, determine the probability that the ensemble classifies correctly using simple majority voting.

a) 0.915
b) 0.8167
c) 0.85
d) 0.9

Answer: a.

Task 22
Intermediate level (5 points)

The neural network shown in the figure below is trained using stochastic gradient descent with a learning rate of $\alpha = 0.1$.



The output neuron's transfer function is linear, and the other neurons have the transfer function defined as:

$$f(h) = \begin{cases} h, & h \geq 0, \\ 0, & h < 0. \end{cases}$$

At the current training step the synaptic coefficients (weights) in the network take the values $v_1 = v_2 = w_1 = w_2 = 1$, $w_3 = -2$, and the biases of all neurons are zero. The network receives a training example $x = 2$ with the target value $\sigma = 3$. Perform a network's training step with a quadratic loss function $L = \frac{1}{2}(\sigma - y)^2$ and calculate the synaptic coefficient v_1 after its adjustment.

Answer: 1.2.

Field of Science 6. Cybernetics

Task 23
Entry level (1 point)

What is the slope of the high-frequency section of the asymptotic logarithmic amplitude-frequency response of the aperiodic link?

a) –20 dB/dec.
b) 0 dB/dec.
c) 20 dB/dec.
d) 40 dB/dec.
e) –40 dB/dec.

Answer: a.

Task 24
Entry level (2 points)

The transfer function of the system is $W(s) = \frac{s^2-2s+4}{s^2-25}$. Which typical dynamic link does it contain?

a) unstable aperiodic
b) first-order differentiator
c) second-order differentiator

d) oscillatory
e) integrating

Answer: a.

Task 25
Entry level (3 points)

Determine the duration of the transition process in a nonlinear system from point A to point D, given that the corresponding section of the phase trajectory on the phase plane xoy (where x is the system's output coordinate and y is its rate of change) consists of straight segments connecting points A (–4;3), B (2;3), C (2;1), and D (6;1).

a) 6
b) 2
c) 4
d) 8

Answer: a.

Task 26
Intermediate level (4 points)

At what values of the parameter a is a closed-loop system with single and negative feedback on the boundary of stability, if the differential equation of the open-loop system is given by the following (where u(t) and y(t) denote the input and output of the open-loop system, respectively):

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} - ay(t) = \frac{d^2u(t)}{dt^2} - a\frac{du(t)}{dt} + 5u(t).$$

Answer: 4.

Field of Science 7. Information Systems

Task 27
Entry level (1 point)

What is the main structural part of the relational model?

a) attitude
b) attribute
c) tuple

Answer: a.

Task 28
Entry level (1 point)

According to the CAP theorem, how many properties can be satisfied in a distributed system?

a) 1

b) 2
c) 3
d) 4

Answer: b.

Task 29
Entry level (2 points)

What are ETL processes used for?

a) to ensure quality, create the necessary structure, and maintain the semantic characteristics of the data
b) to create relational tables
c) to carry out procedures for searching, filtering, and sorting data in the warehouse after receiving an SQL query

Answer: a.

Task 30
Intermediate level (4 points)

Two relations are given—A and B. Relation A has the form {Student ID, Student Last Name, Group, Grade}, and relation B has the form {Student ID, Student Last Name, Student Name, Subject, Grade}. How many attributes will relation C have after performing the operation C = A JOIN B?

Answer: 6.