Mathematics and Artificial Intelligence: second-round sample tasks

1 Low difficulty problems

- [2 points] Calculate scalar and vector products of vectors $a = (1, -1, 2)^T$ and $b = (-3, 1, 1)^T$.
  Answer: -2, $(-3, -7, -2)^T$.
- [2 points] Let
  $$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$
  Find the product of the eigenvalues of A.
  Answer: 27.
- [2 points] Calculate the second derivative of $f(x) = \log(\tan^3(x))$ at $x = \frac{\pi}{6}$. (log is natural logarithm)
  Answer: -8.
- [2 points] Is the function $y(x) = 2e^{2x} + 3$ a solution of Cauchy problem $y' = 2y - 5; \quad y(0) = 5$? Is it a solution of differential equation $y' = 2y - 5$?
  Answer: No; no.
- [2 points] Consider the differential equation $y' = 4y - 3 - y^2$. What is the limit of all its solutions as $x \to +\infty$?
  Answer: 3.
- [2 points] An urn contains 2 blue, 3 red and 4 violet balls. A pair of balls is extracted. What is the probability of both balls being red?
  Answer: $\frac{1}{12}$.
- [2 points] Consider the classification problem with feature matrix
  $$X = \begin{pmatrix} 6 & 3 \\ 2 & 7 \\ 9 & 6 \\ 4 & 2 \end{pmatrix}$$
  and label vector $y = (1, 0, 1, 0)^T$. Classification is done by constructing a decision tree. Denote by f_i the feature corresponding to i-th column of matrix X. Which of follwing divisions in the root vertex provides the largest increase of information?
  1. $f_1 > 2$
  2. $f_1 > 3$
  3. $f_1 > 4$
  4. $f_1 > 6$

  As an answer, write the number of corresponding variant.
  Answer: 3.
- [2 points] Consider the feature matrix $X = \begin{pmatrix} 6 & -4 \\ -3 & 5 \\ -2 & 6 \\ 7 & -3 \end{pmatrix}$. Solve the data dimension reduction problem by main component analysis method. Calculate the covariation matrix. Find the representation of the subspace, on which data are projected by main component analysis method, as a straight line $ax + by + c = 0$. As an answer, enter the sum of all elements of covariation matrix and coefficients a, b, c.
  Answer: 4.

- [2 points] What pairs are comparable modulo 11: (0,11), (2,35), (4,17), (-5, 7), (25,-19).

  Answer: First, second, fifth.

- [2 points] Consider graph G, such that: 1) G is a tree, and 2) G has 20 vertices. What is the maximal and minimal number of endpoints of G?

  Answer: 19, 2.

2 Medium difficulty problems

- [7 points] Find the type of surface given by equation $x^2 + y^2 + z^2 - 4xy - 4xz - 4yz + 24 = 0$.

  Answer: parted (two-sheeted) hyperboloid.

- [7 points] Calculate the integral $\int_0^{2\pi} \frac{dx}{\cos(x)+\sin(x)+2}$.

  Answer: $\sqrt{2}\pi$.

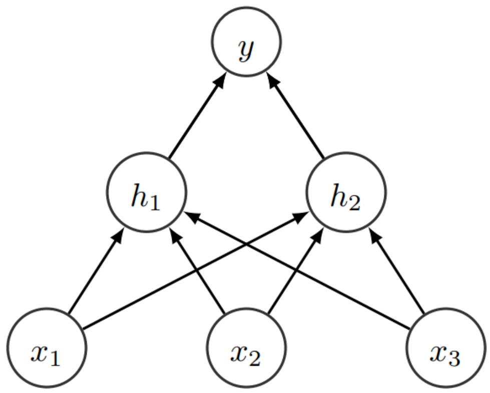
- [7 points] Let ξ be a random variable with density

$$p_\xi(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases} \quad (1)$$

  Find the mode (a point where the density is maximal) of random variable $\eta = (\xi + 1)^2$.

  Answer: 1.

- [7 points] Consider the artificial neural network presented as a figure. Network is provided with the input



  $x = (1, 2, 1)^T$ and output $y = 1$. Network weights are initialized as follows: $W = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $W_2 = (0, 1)$, displacements are equal to 0. Hidden and output layers use activation function $\sigma(z) = \max\{0, z\}$. Denote as $\hat{y}$ the output value of the network. The error function is $l(y, \hat{y}) = \frac{1}{2}(y - \hat{y})^2$. Calculate $\hat{y}$. Calculate the gradient matrix $\frac{\partial l}{\partial W_1}$. Put the sum of $\hat{y}$ and all elements of $\frac{\partial l}{\partial W_1}$ into the answer box.

  Answer: 0.

- [7 points] Which of functions $xy, x \vee y, x \oplus y, 1$ can be excluded from the system without breaking the completeness of the system?

  Answer: $xy, x \vee y$.

3 High difficulty problems

- [15 points] Let u be the solution of Dirichlet problem

$$-u'' = 3x^2; \quad u(0) = u(1) = 0. \tag{2}$$

Find $u'(1) - u'(0)$.

Answer: -1.

Solution:

$$u'(1) - u'(0) = \int_0^1 u''(x)dx = \int_0^1 (-3x^2)dx = -1.$$

- [15 points] Let ξ_1, ξ_2 be two independent exponential random variables with density

$$p_\xi(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases} \tag{3}$$

Find the probability of $\{\xi_1 - 2\xi_2 \in [-2, 1]\}$.

Solution: The characteristic function of ξ_1 is

$$\varphi_{\xi_1}(t) = E(\exp(it\xi_1)) = \int_0^\infty e^{itx}e^{-x}dx = \frac{1}{1-it}.$$

Similarly,

$$\varphi_{-2\xi_2}(t) = E(\exp(-2it\xi_2)) = \int_0^\infty e^{-2itx}e^{-x}dx = \frac{1}{1+2it}.$$

Characteristic function of sum of two independent variables is the product of characteristic functions of summands, so

$$\varphi_{\xi_1-2\xi_2}(t) = \frac{1}{1-it} \cdot \frac{1}{1+2it}.$$

To decompose this expression into partial fractions, we write

$$\frac{1}{1-it} \cdot \frac{1}{1+2it} = \frac{A}{1-it} + \frac{B}{1+2it}.$$

Coefficients A and B are calculated by multiplying the expression by its common denominator and obtaining the identity

$$1 = A(1+2it) + B(1-it).$$

Substituting $t = -i$ and $t = \frac{i}{2}$, we obtain $A = \frac{1}{3}$ and $B = \frac{2}{3}$. Then

$$\varphi_\eta(t) = \frac{1/3}{1-it} + \frac{2/3}{1+2it}.$$

We can now find the density p_η as the inverse Fourier transform of φ_η. Another way of calculating p_η is to observe that:

– characteristic function $\frac{1}{1-it}$ corresponds to density $\mathbf{1}_{[0,+\infty)} \cdot e^{-x}$;

– characteristic function $\frac{1}{1+2it}$ corresponds to density $\mathbf{1}_{(-\infty,0]} \cdot \frac{1}{2}e^{\frac{x}{2}}$.

Then by linearity of inverse Fourier transform we obtain

$$p_\eta(x) = \begin{cases} \frac{1}{3}e^{-x}, & x \geq 0; \\ \frac{1}{3}e^{\frac{x}{2}}, & x < 0. \end{cases}$$

Then

$$P(\{\xi_1 - 2\xi_2 \in [-2, 1]\}) = \int_{-2}^1 p_\eta(x)dx = \int_{-2}^0 \frac{1}{3}e^{\frac{x}{2}}dx + \int_0^1 \frac{1}{3}e^{-x}dx = \frac{2}{3}(1-e^{-1}) + \frac{1}{3}(1-e^{-1}) = 1 - e^{-1}.$$

Answer: $1 - e^{-1}$.

- [15 points] Find all solutions to $24n = 65\varphi(n)$, where $\varphi(n)$ is the Euler function.

  Solution: Decompose n into prime factors: $n = p_1^{k_1} \cdot \ldots \cdot p_m^{k_m}$. Then

$$\varphi(n) = p_1^{k_1-1}(p_1-1) \cdot \ldots \cdot p_m^{k_m-1}(p_m-1).$$

Observe that expression

$$\frac{\varphi(n)}{n} = \frac{p_1-1}{p_1} \cdot \ldots \cdot \frac{p_m-1}{p_m}$$

does not depend on exponents $k_1, \ldots, k_m$. It means that they can have any natural values, and the problem is reduced to finding factors $p_1, \ldots, p_m$.

Consider the identity

$$24p_1 \ldots p_m = 65(p_1-1)\ldots(p_m-1),$$

where $p_1, \ldots, p_m$ are prime numbers. RHS is divisible by 5, so one of prime factors in the LHS must be 5. Without loss of generality assume that $p_1 = 5$. Then

$$24 \cdot 5 \cdot p_2 \ldots p_m = 65 \cdot 4 \cdot (p_2-1)\ldots(p_m-1),$$

from which

$$6 \cdot p_2 \ldots p_m = 13 \cdot (p_2-1)\ldots(p_m-1).$$

RHS is divisible by 13, so one of prime factors in the LHS must be 13. Without loss of generality assume that $p_2 = 13$. Then

$$6 \cdot 13 \cdot p_3 \ldots p_m = 13 \cdot 12 \cdot (p_3-1)\ldots(p_m-1).$$

After simplification we have

$$p_3 \ldots p_m = 2 \cdot (p_3-1)\ldots(p_m-1).$$

RHS is divisible by 2, so without loss of generality we can assume that $p_3 = 2$. Dividing the equation by 2, we obtain $1 = 1$, if $m = 3$, and

$$p_4 \ldots p_m = (p_4-1)\ldots(p_m-1),$$

if $m \geq 4$. LHS in the last identity is strictly greater than RHS, so the initial equation has no solutions with $m \geqslant 4$. Finally, $n = 5^{k_1} \cdot 13^{k_2} \cdot 2^{k_3}$, where $k_1, k_2, k_3 \in \mathbb{N}$.